

VIII. *An Account of a Book, intituled, Harmonia Mensurarum, five Analysis & Synthesis per Rationum & Angulorum mensuras promotæ : accedunt alia Opuscula Mathematica : per Rogerum Cotesium. Edidit & auxit Robertus Smith, Coll. Trin. Cantab. & Reg. Soc. Socius; Astronomiæ & Experimentalis Philosoph. post Cotesium Professor. Cantabrigiæ 1722. in 4to. Prostant apud Bibliopolas Londinenses.*

THE Book consists of three Parts. In the first, called *Logometria*, the Author's chief Design is to shew how that sort of Problems, which are usually reduced to the Quadrature of the *Hyperbola* and *Ellipsis*, may be reduced to the Measures of Ratio's and Angles; and afterwards be solved more readily by the Canons of Logarithms and Sines and Tangents. He defines the Measures of Ratio's to be quantities of any kind, whose Magnitudes are analogous to the Magnitudes of the Ratio's to be measured. In this Sense any Canon of Logarithms is a System of numeral Measures of the Ratio's of the absolute Numbers to an Unit: The Parts of the Asymptote of the Logistic Line, intercepted between its Ordinates, are a System of Linear Measures of the Ratio's of those Ordinates: The Areas of an *Hyperbola*, intercepted between its Ordinates to the Asymptote, are a System of Plane Measures of the Ratio's of those Ordinates: And since there may be infinite Systems of Measures according
as

as various kinds of Quantities are made use of, such as Numbers, Time, Velocity, and the like ; or according as the Measures of any one System may be all increased or diminished in any given proportion ; in such Variety much Confusion may possibly arise as to the Kind and absolute Magnitudes of particular Measures, which happen to fall under Consideration. Our Author very happily removes this Difficulty ; by shewing that the Nature of the Subject points out the Measure of a certain immutable Ratio for a *Modulus* in all Systems, whereby to determine the Kind and absolute Magnitudes of all other Measures in each System.

The first Proposition is to find the Measure of any proposed Ratio. This he considers in a way so simple and general, as naturally leads to the Notion and Definition of a *Modulus* ; namely, that it is an invariable Quantity in each System, which bears the same Proportion to the Increment of the Measure of any proposed Ratio, as the increasing Term of the Ratio bears to its own Increment. He then shews, that the Measure of any given Ratio is as the *Modulus* of the System, from whence it is taken : and that the *Modulus* in every System is always equal to the Measure of a certain determinate and immutable Ratio, which he therefore calls the *Ratio Modularis*. He shews that this Ratio is expressed by these Numbers 2,7182818 *&c.* to 1, or by 1 to 0,3678794 *&c.* So that in *Briggs's Canon* the Logarithm of this Ratio is the *Modulus* of that System : In the Logistic Line the given Subtangent is the *Modulus* of that System : In the *Hyperbola* the given Parallelogram, contained by an Ordinate to the Asymptote and the Absciss from the Center, is the *Modulus* of that System : and in other Systems the *Modulus* is generally some remarkable Quantity. In the second Proposition he gives a

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concise

concise uncommon Method for calculating *Briggs's* Canon of Logarithms ; together with Rules for finding intermediate Logarithms and Numbers, even beyond the Limits of the Canon. In the 3d Proposition he constructs any System of Measures by a Canon of Logarithms; not only when the Measure of some one Ratio is given, but also without that *Datum*, by seeking the *Modulus* of the System by the Rule abovementioned. In the 4th, 5th, and 6th Propositions he squares the *Hyperbola*, describes the Logistic Line and Æquiangular Spiral by a Canon of Logarithms, and shews some curious Uses of these Propositions in their *Scholia*. Take an easy Example of the Logometrical Method, in the common Problem for finding the Density of the Atmosphære. Supposing Gravity uniform, every one knows, that if Altitudes are taken in any Arithmetical Progression, the Densities of the Air in those Altitudes will be in a Geometrical Progression ; that is, the Altitudes are the Measures of the Ratio's of the Densities below and in those Altitudes, and so the difference of any two Altitudes is the Measure of the Ratio of the Densities in those Altitudes. Now to determine the absolute or real Magnitude of these Measures, the Author shews, *a priori*, that the *Modulus* of the System is the Altitude of the Atmosphære, when reduced every where to the same Density as below. The *Modulus* therefore is given (as bearing the same Proportion to the Altitude of the Mercury in the Barometer, as the specific Gravity of Mercury does to the specifick Gravity of Air) and consequently the whole System is given. For since in all Systems the Measures of the same Ratio's are analogous among themselves ; the Logarithm of the Ratio of the Air's Density in any two Altitudes will be to the *Modulus* of the Canon, (that is, to the Logarithm of

the *Ratio Modularis* defined above,) as the difference of those Altitudes is to the aforesaid given Altitude of the homogeneous Atmosphære.

He concludes the Logometria with a *General Scholium*, containing great Variety of elegant Constructions both Logometrical and Trigonometrical ; such as give the Length of Curves either Geometrical or Mechanical ; their Area's and Centers of Gravity ; the Solids generated from them, and the Surfaces of these Solids ; together with several curious Problems in Natural Philosophy, concerning the Attraction of Bodies, the Density and Resistance of Fluids, and the Trajectories of Planets. Several of these Problems have two Cases ; one constructed by the Measure of a Ratio, and the other by the Measure of an Angle. The great Affinity and beautiful *Harmony* of the *Measures* in these Cases, has given occasion to the Title of the Book. The Measures of Angles are defined (just as the Measures of Ratio's) to be Quantities of any Kind, whose Magnitudes are analogous to the Magnitudes of the Angles. Such may be the Arcs or Sectors of any Circle, or any other Quantities of Time, Velocity, or Resistance, analogous to the Magnitudes of the Angles. Every System of these Measures has likewise its *Modulus* homogeneous to the Measures in that System, and may be computed by the Trigonometrical Canon of Sines and Tangents, just as the Measures of Ratio's by the Canon of Logarithms ; for the given *Modulus* in each System bears the same Proportion to the Measure of any given Angle, as the *Radius* of a Circle bears to an Arc which subtends that Angle, or the same as this constant Number of Degrees 57,2957795130 bears to the Number of Degrees in the said Angle. Upon the whole our Author thus expresses himself, *p.* 35. “ Ex
“ adductis

“ adductis hæcenus exemplis, Geometris integrum erit
 “ de methodo nostrâ judicare ; quam quidem, si pro-
 “ ba fuerit, ulterius excolere pergent & excolendo la-
 “ tius promovebunt. Patet utique campus amplissi-
 “ mus in quo vires suas experiri poterunt, præsertim si
 “ Logometriæ Trigonometriam insuper adjungant,
 “ quibus miram quandam affinitatem in se invicem
 “ euntibus intercedere notabam. Hisce quidem prin-
 “ cipiis haud facile crediderim *generaliora* dari posse ;
 “ cum tota Mathesis vix quicquam in universo suo am-
 “ bitu complectatur præter Angulorum & Rationum
 “ Theoriam. Neque sane *commodiora* sperabit, qui
 “ animadverterit effectationis facilitatem per amplissi-
 “ mas illas, omnibusque suis numeris absolutas, tum
 “ Logarithmorum, tum Sinuum & Tangentium tabu-
 “ las ; quas antecessorum nostrorum laudatissimæ
 “ solertiæ debemus acceptas. Ut vero tanti beneficii
 “ uberior nobis exfurgat fructus, id nunc exponendum
 “ restat, quibus artibus ad istius modi conclusiones re-
 “ ctissimâ perveniatur. In hunc finem *Theoremata*
 “ quædam tum *Logometrica* tum *Trigonometrica* ad-
 “ jecissem, quæ parata ad usum asservo ; ni consul-
 “ tius visum esset, quum absque nimis ambagibus ea
 “ tradi non possent, intacta potius præterire atque
 “ aliis denuo investiganda relinquere.

Why the Author takes his Principles to be so gene-
 ral, will farther appear by an Instance or two. In the
 Problem already mentioned he measures the Ratio of
 the Air's Densities in any Altitudes, by the Altitudes
 themselves, making use of the Altitude of an uniform
 Atmosphere for the *Modulus*. So likewise when he
 considers the Velocities acquired, and the Spaces de-
 scribed in given Times, by a Body projected upwards
 or downwards in a resisting Medium with any given
 Velocity ; he shews, that the Times of Descent, added

to a given Time, are the Measures of Ratio's, to a given *Modulus* of Time, whose Terms are the Sum and Difference of the ultimate Velocity and the present Velocities that are acquired : that the Times of Ascent, taken from a given Time, are the Measures of Angles, to a given *Modulus* of Time, whose Radius is to their Tangents in the Ratio of the ultimate Velocity to the present Velocities : and lastly, that the Spaces described in Descent or Ascent, are the Measures of Ratio's to a given *Modulus* of Space, whose Terms are the absolute accelerating and retarding Forces arising from Gravity and Resistance taken together at the Beginning and End of those Spaces.

This general Account may suffice to illustrate what I am going to say ; that since the Magnitudes of Ratio's (as well as their Terms) may be expounded by Quantities of any Kind, the Mathematician is at Liberty upon all Occasions to chuse those which are fittest for his Purpose ; and such are they without doubt, that are put into his Hand by the Conditions of the Problem. He may indeed represent these Quantities by an *Hyperbola*, or any other Logometrical System, were not his Purpose answer'd with greater Simplicity by the very System itself, which occurs in each particular Problem. And the same may be said for the Systems of Angular Measures, instead of recurring upon all Occasions to Elliptical or Circular Area's.

As to the Convenience of calculating from our Author's Constructions, he shews that the Measures of any Ratio's or Angles are always computed in the same uniform Way ; by taking from the Tables the Logarithm of the Ratio, or the Number of Degrees in the Angle, and then by finding a fourth Proportional to three given Quantities ; for that will be the Measure required

required. The simplest Hyperbolic Area may indeed be squared by the same Operation taught in the Author's fourth Proposition; but the simplest Elliptic Area requires somewhat more: Those that are more complex in both Kinds (which generally happens) require an additional Trouble to reduce them to the simplest: to square them by infinite Series is still more operose, and does not answer the End of Geometry. Upon the whole therefore it may deserve to be considered, for what Purposes should Problems be always constructed by Conic Areas, unless it be to please or assist the Imagination. The Design of Theoretical Geometry differs from Problematical; the former consists in the Discovery and Contemplation of the Properties and Relations of Figures for the sake of naked Truth; but the Design of the latter is to do something proposed, and is best executed by the least *Apparatus* of the former.

The *Logometria* was first published by the Author himself, in the *Philosoph. Transact.* of the Year 1714. No 338. But his Logometrical and Trigonometrical Theorems abovementioned were not published till after his Decease. These Theorems make the second Part of the Book, and are calculated to give the Fluents of Fluxions (reduced to 18 Forms) by Measures of Ratio's and Angles; in such a manner, that any Person may perfectly comprehend their Construction and Use, though altogether unacquainted with Curvilinear Figures, as expressed by *Æquations*. And this Circumstance does also render the Application of them to the *Analysis* and *Construction* of *Problems* extremely easy. Of this kind the Author has given a great many choice Examples both in abstract and physical Problems; which make up the third and last Part of the Book.

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The Author, a little before his Decease has informed us (in a Letter of *May 5.* wrote to his Friend Mr. *Jones*)
 “ that Geometers had not yet promoted the inverse
 “ Method of Fluxions, by Conic Areas, or by Mea-
 “ sures of Ratio's and Angles, so far as it is capable of
 “ being promoted by those Methods. There is an
 “ infinite Field (says he) still reserved, which it has
 “ been my Fortune to find an Entrance into. Not to
 “ keep you longer in Suspense, I have found out a ge-
 “ neral and beautiful Method by Measures of Ratio's
 “ and Angles for the Fluent of any Quantity which

“ can come under this Form $\frac{dz z^{\theta n + \frac{\delta}{\lambda} n - 1}}{e + fz^n}$, in

“ which d, e, f are any constant Quantities, z the vari-
 “ able, n any Index, θ any whole Number affirmative or

“ negative, $\frac{\delta}{\lambda}$ any Fraction whatever. The Fluents of
 “ this Form which have hitherto been considered are

$\frac{dz z^{\theta n - 1}}{e + fz^n}$ & $\frac{dz z^{\theta n + \frac{1}{2} n - 1}}{e + fz^n}$: These you remem-

“ ber are Sir *Isaac Newton's* two first, and from
 “ these all his others are easily deduced. And as his
 “ irrational Forms of the quadratic Kind are derived
 “ from the rational, so from my general rational
 “ Form I deduce irrational ones of all Kinds. For in-

“ stance, if $\frac{\delta}{\lambda}$ represent any affirmative or negative
 “ Fraction, the Fluent of any Quantity of this Form

“ $dz z^{\theta n - 1} \times \sqrt[\delta]{e + fz^n}$, or of this $dz z^{\theta n - 1}$

“ $\times \sqrt[\delta]{\frac{e + fz^n}{g + bz^n}}$ and so of some others, depends upon the

“ the Measures of Ratio's and Angles. Mr. *Leibnitz*
 “ in the *Leipſic Acts* of 1702, p. 218 and 219, has ve-
 “ ry raſhly undertaken to demonſtrate, that the Fluent
 “ of $\frac{x}{x^4+a^4}$ cannot be expreſſed by Measures of Ra-
 “ tio's and Angles ; and he ſwaggers upon the Occa-
 “ ſion (according to his uſual Vanity) as having by
 “ this Demonſtration determined a Queſtion of the
 “ greateſt Moment. Then he goes on thus ; as the
 “ Fluent of $\frac{x}{x+a}$ depends upon the Measure of a
 “ Ratio, and the Fluent of $\frac{x}{xx+aa}$ upon the Measure
 “ of an Angle ; ſo he had more than once expreſſed
 “ his Wiſhes, that the Progreſſion may be continued,
 “ and it be determined to what Problem the Fluents of
 “ $\frac{x}{x^4+a^4}$, $\frac{x}{x^8+a^8}$, &c. may be referred. His De-
 “ ſire is answered in my general Solution, which
 “ contains an infinite Number of ſuch Progreſſions.
 “ I can go yet farther, and ſhew him how by Mea-
 “ ſures of Ratio's and Angles, without any Exception
 “ or Limitation, the Fluent of this general Quantity
 “ $\frac{dz z^{\theta n + \frac{\delta}{\lambda} n - 1}}{e + f z^n + g z^{2n}}$ or even this $\frac{dz z^{\theta n + \frac{\delta}{\lambda} n - 1}}{e + f z^n + g z^{2n} + h z^{3n}}$
 “ may be had ; where θ , as before, represents any Inte-
 “ ger, and the Denominator λ of the Fraction $\frac{\delta}{\lambda}$, re-
 “ presents any Number in this Series, 2. 4. 8. 16. 32. &c.
 “ any whole Number being denoted by its Numerator
 “ δ . In truth I am inclined to believe, that Mr. *Leib-*
 “ *nitz's* grand Queſtion ought to be determined
 “ the

“ the contrary Way ; and that it will be found at laſt,
 “ that the Fluent of any rational Fluxion whatever,
 “ does depend upon the Meaſures of Ratio's and
 “ Angles, excepting thoſe which may be had in finite
 “ Terms even without introducing Meaſures.

Dr. *Taylor* knowing by this Letter what the Author had done, was pleaſed to propoſe the Invention of the Fluents of the two laſt Fluxions as a Problem to the Mathematicians in foreign Parts. Mr. *Bernoulli* in the *Leipſic Acts* of 1719. p. 256, did ſhew accordingly how they are reducible to Conic Area's. The Editor has publiſhed the Author's own Solution by Meaſures of Ratio's and Angles ; and upon this Foundation has conſtructed new Tables of Logometrical and Trigonometrical Theorems, for the Fluents of Fluxions reduced to 94 Forms, part rational and part irrational. He has likewiſe added general Notes upon the chief Difficulties in the Book, together with a Method of compoſing Synthetical Demonſtrations of Logometrical and Trigonometrical Conſtructions, illuſtrated by various Examples.

The firſt Treatiſe in the *Miſcellaneous Works* is concerning the *Eſtimation of Errors in Mixt Mathematicks*. It conſiſts of 28 Theorems, to determine the Proportions among the leaſt contemporary Variations of the Sides and Angles of Plane and Sphærical Triangles, while any two of them remain invariable. An Example will ſhew their great Uſe in Aſtronomy. The Time of the Day or Night is frequently to be determined by the Altitude of ſome Star. Let it then be propoſed to find the Error, that may ariſe in the Time, from any given Error in taking the Altitude. By applying the 22d Theorem to the Triangle form'd by the Complements of the Star's Altitude and Declination
 and

and by the Complement of the Pole's Elevation, the Author shews, that the Variation of the Angle at the Pole, and consequently the Error in Time, will be as the Error in the Altitude directly, as the Sine Complement of the Pole's Elevation inversely, and as the Sine of the Star's *Azimuth* from the Meridian inversely. Consequently, if the Error in the Altitude be given, under a given Elevation of the Pole, the Error in Time will be reciprocally as the Sine of the *Azimuth* contained by the Meridian and the Vertical which the Star is in. This Error therefore will be the same, whatever be the Altitude of the Star in the same Vertical; and will be least when the Vertical is at right Angles to the Meridian. But will be absolutely the least in the same Circumstance, if the Observer be under the *Æquator*. In which Case, if the Error in the Altitude be one Minute, the Error in the Time will be four Seconds; If the Observer recedes from the *Æquator* towards either Pole, the Error will be increased in the Proportion of the Radius to the Sine Complement of the Latitude: So that in the Latitude of 45 Degrees it will be $5\frac{1}{2}$ Seconds, and in the Latitudes of 50 and 55 it will be $6\frac{1}{2}$ and $6\frac{3}{4}$ Seconds respectively. If the Star be in any other vertical Oblique to the Meridian, the Error will still be increased in the Proportion of the Radius to the Sine of that oblique Angle. Lastly, if the Error in the Altitude be either bigger or less than one Minute, the Error in Time will be bigger or less in the same Proportion. Much after the same manner may the Limits of Errors be computed in other Cases, which arise from the Inaccuracy of Observations, and from hence the most convenient Opportunities for observing are also determined.

The Second Treatise is concerning the *Differential Method*. The Author having wrote it, before he had

seen Sir *Isaac Newton's* Treatise upon that Subject, has handled it after a manner somewhat different.

The Title of the Third Treatise is *Canonotechnia* or concerning the Construction of Tables by Differences: It consists of ten Propositions, most admirably contrived for expeditious Computation of intermediate Terms in any given Series. The last Proposition, which contains a general Solution of the whole Design, is this; *Datis seriei cujuscunque terminis aliquot equidistantibus, quorum intervalla secunda sunt in aequales quotlibetcunq; partes, propositum sit invenire terminos interserendos.*

The Book concludes with three small Tracts, concerning the Descent of Bodies, the Motion of Pendulums in the Cycloid, and the Motion of Projectiles, composed in a very natural and easy manner.

The Author has wrote some other Pieces, yet unpublsh'd, which the Editor has given an Account of in his Preface to the Book.

The Reader will find every Subject treated with uncommon Elegance and Simplicity.

F I N I S.

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